Q-boson representation of the quantum matrix algebra $M_q(3)$

1993 J. Phys. A: Math. Gen. 26 L957

(http://iopscience.iop.org/0305-4470/26/18/013)

J. Phys. A: Math. Gen. 26 (1993) L957–L962. Printed in the UK

LETTER TO THE EDITOR

Q-boson representation of the quantum matrix algebra $M_q(3)$

Vahid Karimipour

Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395–1795, Tehran, Iran

Department of Physics, Sharif University of Technology P.O. Box 11365–9161, Tehran, Iran

Received 12 February 1993

Abstract. Although q-oscillators have been used extensively for realization of quantum universal enveloping algebras, such realizations do not exist for quantum matrix algebras (deformation of the algebra of functions on the group). In this letter we first construct an infinite dimensional representation of the quantum matrix algebra $M_q(3)$ (the coordinate ring of $GL_q(3)$) and then use this representation to realize $GL_q(3)$ by q-bosons.

Since the advent of q-oscillators or q-boson algebras [1–3] a lot of attention has been paid to realization of quantum universal enveloping algebras (QUEA) [4–8] in terms of q-oscillators. However, the corresponding task for the dual objects, i.e. quantum matrix algebras, has not been studied so far, except for the case of $GL_q(2)$ [9]. In this paper we extend the results of [9] and give a three-parameter family of q-boson realization for the quantum matrix algebra $M_q(3)$.

The quantum algebra $GL_q(3)$ is generated by the elements of a matrix

$$T=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \tag{1}$$

subject to the relations

$$RT_1T_2=T_2T_1R \tag{2}$$

where R is the solution of the Yang-Baxter equation corresponding to $SL_q(3)$ [5].

$$R=\sum_{i\neq j}^{n} e_{ii}\otimes e_{jj}+\sum_{i=1}^{n} qe_{ii}\otimes e_{ii}+(q-q^{-1})\sum_{i<j}^{n} e_{ji}\otimes e_{ij} \tag{3}$$

The relations obtained from (2) can be expressed neatly in the following form: for any 2×2 submatrix (i.e. like the one formed by the elements b, c, e, and f) the following relations hold:

$$\begin{aligned} &bc=qcb && ef=qfe \\ &be=qeb && cf=qfc \\ &ec=ce && bf-fb=(q-q^{-1})ce. \end{aligned} \tag{4}$$

Remark. These relations are only a small part of the relations obtained from (2). All the other relations can be simply read by looking at other submatrices. (i.e. $df = qfd$, $dg = qgd$, $dk - kd = (q - q^{-1})fg$, etc.). Hereafter when we refer to (4) we mean all the relations of which (4) is a sample. Thus this algebra has many $GL_q(2)$ subalgebras (i.e. the elements a, c, d, and f generate a $GL_q(2)$ subalgebra). Obviously these are not Hopf subalgebras.

Remark. One can prove the following more general type of formula

$$bf^n - f^n b = q^{-1}(q^{2n} - 1)f^{n-1}ce \tag{5}$$

by induction from (4).

$GL_q(3)$ has also a quantum determinant D [8] which is central:

$$D = a\Delta_a - qb\Delta_b + q^2 c\Delta_c \tag{6}$$

where Δ_a, Δ_b and Δ_c are the quantum cofactors of the elements a, b and c respectively:

$$\Delta_a = ek - qfh \qquad \Delta_b = dk - qfg \qquad \Delta_c = dh - qeg. \tag{7}$$

One can also see from (4) that the elements c, e and g commute with each other, a fact which will play an important role in building up the representation. It is clear that the eigenvalues of the operators c, e and g will label the states of any representations. What remains to be done is the choice of lowering and raising operators. At first sight one may try to choose the operators f, h and k as raising and a, b and d as lowering operators, and construct a Verma module out of the states $|l, m, n\rangle \equiv f^l h^m k^n|0\rangle$ where $|0\rangle$ is the vacuum which is an eigenstate of c, e and g and satisfies: $a|0\rangle = b|0\rangle = d|0\rangle = 0$. But this choice has the disadvantage that to compute the action of a lowering operator like a on $|l, m, n\rangle$ one must use commutation relations of the type (5) many times, which makes the computation cumbersome and the results not illuminating. However, a much better approach is possible, which we now explain.

We first *construct an infinite dimensional representation of a subalgebra of* $M_q(3)$. This subalgebra which we will denote by A is generated D and all the elements of the matrix T (except a and k) plus two quantum cofactors Δ_a and Δ_k. Hereafter we denote them by Δ and Δ' respectively:

$$\Delta = ek - qfh \qquad \Delta' = ae - qbd. \tag{8}$$

As we will see the elements Δ and Δ' rather than k and a will be the natural choice of the third pair of raising and lowering operators. The first two pairs are (f, b) and (h, d). Clearly the element Δ being the q-determinant of the submatrix

$$\begin{pmatrix} e & f \\ h & k \end{pmatrix}$$

commutes with the elements e, f, h and k. A similar statement holds true for Δ'. (i.e. Δ' commutes with a, b, d and e). Using (4, 8) it is straightforward to verify the following commutation relations:

$$b\Delta = q\Delta b \qquad c\Delta = q\Delta c$$
$$d\Delta = q\Delta d \qquad g\Delta = q\Delta g \tag{9}$$

and

$$c\Delta' = q^{-1}\Delta' c \qquad g\Delta' = q^{-1}\Delta' g$$
$$f\Delta' = q^{-1}\Delta' f \qquad h\Delta' = q^{-1}\Delta' h. \tag{10}$$

We need two other relations which we present below:

Lemma.
(i) $a\Delta = q^2\Delta a + (1-q^2)D$ (11)

(ii) $\Delta'\Delta = q^2\Delta\Delta' + (1-q^2)De.$ (12)

Where D is the quantum determinant of the matrix T (see (6)).

Proof. (i) Passing a through Δ and using the commutation relations (4) we find

$$a\Delta = \Delta a + (q-q^{-1})(ecg + bdk - qfbg - qcdh)$$

and from (4) we have

$$bdk - qfbg = bdk - q(bf - (q-q^{-1})ce)g = b\Delta_b + (q^2-1)ceg$$

therefore the sum of the terms in the bracket is equal to $b\Delta_b - qc\Delta_c$ and hence the above relation is transformed to the following form:

$$a\Delta = \Delta a + (q-q^{-1})(b\Delta_b - qc\Delta_c).$$

By using the expression of the quantum determinant (6) we finally arrive at (11).

The proof of (ii) is straightforward. One only needs the result of part (i) and equations (9) and (10).

Corollary.
(i) $a\Delta^n = q^{2n}\Delta^n a + (1-q^{2n})D\Delta^{n-1}$

(ii) $\Delta'\Delta^n = q^{2n}\Delta^n\Delta' + (1-q^{2n})De\Delta^{n-1}.$

These formulae are proved by induction from formulae (11) and (12). One must use the commutativity of e and Δ and the fact that D is central. We now construct an infinite dimensional representation of A. Let us denote by $|0\rangle$ a common eigenvector of c, e, and g which is annihilated by the lowering operators

$$b|0\rangle = d|0\rangle = \Delta'|0\rangle = 0 \tag{15}$$

$$c|0\rangle = \lambda|0\rangle \qquad e|0\rangle = \mu|0\rangle \qquad g|0\rangle = \nu|0\rangle. \tag{16}$$

Then we construct the q-analogue of Verma module as follows:

$$W = \{|l, m, n\rangle \equiv f^l h^m \Delta^n|0\rangle \qquad l, m, n \geqslant 0\}. \tag{17}$$

The vectors of this Verma module are eigenvectors of c, e and g:

$$\begin{aligned} c|l, m, n\rangle &= q^{l+n}\lambda|l, m, n\rangle \\ e|l, m, n\rangle &= q^{l+m}\mu|l, m, n\rangle \\ g|l, m, n\rangle &= q^{m+n}\nu|l, m, n\rangle. \end{aligned} \tag{18}$$

Since f, h and Δ commute with each other we obtain

$$\begin{aligned} f|l, m, n\rangle &= |l+1, m, n\rangle \\ h|l, m, n\rangle &= |l, m+1, n\rangle \\ \Delta|l, m, n\rangle &= |l, m, n+1\rangle. \end{aligned} \tag{19}$$

The action of the lowering operators are determined by using (9) and (10). The result is presented in the following.

Theorem.

$$\begin{aligned} &\text{(i)}\ \ b|l, m, n\rangle = q^{m+n-1}(q^{2l}-1)\lambda\mu|l-1, m, n\rangle \\ &\text{(ii)}\ \ d|l, m, n\rangle = q^{l+n-1}(q^{2m}-1)\mu\nu|l, m-1, n\rangle \\ &\text{(iii)}\ \ \Delta'|l, m, n\rangle = q^{l+m}(1-q^{2n})\mu\eta|l, m, n-1\rangle \end{aligned} \tag{20}$$

where η is the value of the quantum determinant in the representation $D|l, m, n\rangle = \eta|l, m, n\rangle$.

Proof. We only prove (iii). The other two parts are similar. From (9) and (10) we have

$$\begin{aligned} \Delta'|l, m, n\rangle &= \Delta' f^l h^m \Delta^n|0\rangle = q^{l+m} f^l h^m \Delta' \Delta^n|0\rangle \\ &= q^{l+m} f^l h^m (q^{2n}\Delta^n\Delta' + (1-q^{2n}) De\Delta^{n-1})|0\rangle = q^{l+m}(1-q^{2n})\mu\eta|l, m, n-1\rangle \end{aligned}$$

where we have used the commutativity of e and Δ and the centrality of D. Equations (18–20) show that W is an infinite dimensional A module. Once one has a representation of the subalgebra A, it is then an easy task to determine the representation of $M_q(3)$ itself. One requires only to determine the action of the generators a and k on $|l, m, n\rangle$.

Theorem.

(i) $k|l,m,n\rangle = q^{-(l+m)}\mu^{-1}(|l,m,n+1\rangle + q^{-1}|l+1,m+1,n\rangle)$ (21)

(ii) $a|l,m,n\rangle = (1-q^{2n})\eta|l,m,n-1\rangle + \mu\nu\lambda(q^{2m}-1)(q^{2l}-1)q^{2n-2}|l-1,m-1,n\rangle.$ (22)

Proof. From (19) we have

$$\Delta|l,m,n\rangle = |l,m,n+1\rangle.$$

Using the fact that Δ has an equivalent description, namely $\Delta = ke - q^{-1}fh$ we find

$$(ke - q^{-1}fh)|l,m,n\rangle = |l,m,n+1\rangle$$

or

$$q^{l+m}\mu k|l,m,n\rangle - q^{-1}|l+1,m+1,n\rangle = |l,m,n+1\rangle$$

which proves (i).

For (ii) we use a similar method:

$$\Delta'|l,m,n\rangle = (ae - qbd)|l,m,n\rangle$$

from which we obtain using (20)

$$\begin{aligned} q^{l+m}(1-q^{2n})\mu\eta|l,m,n-1\rangle &= q^{l+m}\mu a|l,m,n\rangle \\ &\quad - q(q^{l+n-1}(q^{2m}-1)\mu\nu q^{m+n-2}(q^{2l}-1)\lambda\mu|l-1,m-1,n\rangle). \end{aligned}$$

Rearranging the terms gives (22).

Remark. η is not an independent parameter. One can determine its value by acting on any state with D. The result is $\eta = -q^{-3}\lambda\mu\nu$.

Having constructed the infinite dimensional representation we are now ready to realize the generators of the quantum matrix group by q-bosons. We proceed along the lines proposed in [9]. Let us denote by B_q the q-Boson algebra generated by elements a and $a^\dagger$ with the relations

$$\begin{aligned} &aa^\dagger - q^{\pm 1}a^\dagger a = q^{\mp N} \\ &q^{\pm N}a = q^{\mp 1}aq^{\pm N} \\ &q^{\pm N}a^\dagger = q^{\pm 1}a^\dagger q^{\pm N} \end{aligned} \tag{23}$$

We consider the algebra $B_q^{\otimes 3}$ generated by three commuting q-bosons and its natural representation on the q-Fock space F_q

$$\begin{aligned} &\|l,m,n\rangle\rangle \equiv (a_1^\dagger)^l(a_2^\dagger)^m(a_3^\dagger)^n|0\rangle\rangle \\ &a_1\|l,m,n\rangle\rangle = [l]_q\|l-1,m,n\rangle\rangle \\ &a_2\|l,m,n\rangle\rangle = [m]_q\|l,m-1,n\rangle\rangle \\ &a_3\|l,m,n\rangle\rangle = [n]_q\|l,m,n-1\rangle\rangle \\ &q^{N_i}\|0\rangle\rangle = \|0\rangle\rangle \qquad i=1,2,3. \end{aligned} \tag{24}$$

If Ψ is the natural isomorphism from W to F_q satisfying

$$\Psi: |l, m, n\rangle \to ||l, m, n\rangle\rangle$$

then the induced representation Ψ is defined by

$$\Psi^*(g) = \Psi \circ g \circ \Psi^{-1} \qquad \forall g \in \mathrm{End}\, W. \tag{25}$$

We will then find the following three-parameter family of q-boson realization of A.

$$\begin{aligned}
&f \equiv a_1^\dagger \qquad h \equiv a_2^\dagger \qquad \Delta \equiv a_3^\dagger \\
&c \equiv \lambda q^{N_1+N_3} \qquad e \equiv \mu q^{N_1+N_2} \qquad g \equiv \nu q^{N_2+N_3} \\
&b \equiv (q-q^{-1})\lambda\mu q^N a_1 \\
&d \equiv (q-q^{-1})\mu\nu q^N a_2 \\
&\Delta' \equiv -q(q-q^{-1})\mu\eta q^N a_3
\end{aligned} \tag{26}$$

where $N = N_1 + N_2 + N_3$. By using the expression (8) for Δ and Δ' we find the realizations of a and k:

$$k \equiv \mu^{-1} q^{-(N_1+N_2)} (a_3^\dagger + q a_1^\dagger a_2^\dagger) \tag{27}$$

$$a \equiv (q-q^{-1})\lambda\mu\nu q^{N_3} (q^{-2} a_3 + q^N (q-q^{-1}) a_1 a_2). \tag{28}$$

By straightforward manipulations one can verify directly that the elements $a, b, \ldots, k$ defined as above satisfy the commutation relations of the quantum matrix algebra $M_q(3)$.

Similar methods [10] have been used for studying the irreducible* representations of twisted SU(3) group. However, the method of labeling the states and specially the particular choice of raising and lowering operators that we have adopted (equations (8–10)) are completely different from that of [10]. With this choice the problem of classification of all finite dimensional irreducible representations of 3-by-3 quantum matrix groups simplifies considerably (see [11] for the case of $GL_q(3)$). It may be interesting to do the same thing for twisted SU(3).

References

[1] Biedenharn L G 1989 *J. Phys. A: Math. Gen.* **22** L873
[2] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4551
[3] Sun C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983 (1989).
[4] Drinfeld V G 1986 *Proceeding of the ICM* Berkeley, CA p. 798
[5] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247
[6] Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1989 *Alg. Anal.* **1** 178 (in Russian)
[7] Woronowicz S L 1987 *Commun. Math. Phys.* **111** 613; 1990 *Commun. Math. Phys.* **130** 387
[8] Manin Y 1988 CRM *Preprint*
[9] Ge M L, Liu X F and Sun C P 1992 *J. Math. Phys.* **38** 2541
[10] Bragiel K 1989 *Lett. Math. Phys.* **17** 37–44
[11] Karimipour V 1993 *Lett. Math. Phys.* to appear